Metrics of positive Ricci curvature on bundles

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Abstract

We construct new examples of manifolds of positive Ricci curvature which, topologically, are vector bundles over compact manifolds of almost nonnegative Ricci curvature. In particular, we prove that if $E$ is the total space of a vector bundle over a compact manifold of nonnegative Ricci curvature, then $E \times \mathbb{R}^p$ admits a complete metric of positive Ricci curvature for all large $p$.

1 Introduction

According to the soul theorem of J. Cheeger and D. Gromoll, a complete open manifold of nonnegative sectional curvature, denoted $K \geq 0$, is the total space of a vector bundle over a compact manifold with $K \geq 0$. Manifolds of nonnegative Ricci curvature are much more flexible, and nowadays there are many examples of complete manifolds of $\text{Ric} \geq 0$ which are not even homotopy equivalent to complete manifolds of $K \geq 0$. These include manifolds not homotopy equivalent to closed manifolds [GM85], manifolds not satisfying Gromov’s Betti numbers estimate [SY89], manifolds of infinite topological type (see [Men00] and references therein), manifolds with not virtually-abelian fundamental group [Wei88], compact spin Ricci-flat 4-manifolds with nonzero $\hat{A}$-genus [Bes87, 6.27], [Lot00], and complements to certain smooth divisors in compact Kähler manifolds [TY91].

In [BW02] the authors constructed the first (to our knowledge) examples of complete manifolds with $\text{Ric} > 0$ which are homotopy equivalent but not homeomorphic to complete manifolds with $K \geq 0$. Topologically, the manifolds are vector bundles over tori (in [BW02] we also constructed vector bundles over nilmanifolds carrying $\text{Ric} > 0$). According to [ÖW94, BK01, BK03] in each rank only finitely many vector bundles over tori admit metrics with $K \geq 0$, and more generally, a majority of vector bundles over a fixed manifold $B$ with $K \geq 0$ does not admit a metric with $K \geq 0$, provided $B$ has a sufficiently large first Betti number.

In this paper we greatly extend the results of [BW02]. In particular, in [BW02] we asked whether most vector bundles over compact manifolds with $\text{Ric} \geq 0$ admit metrics with $\text{Ric} > 0$, and in Corollary 1.2 we show that this is true stably, i.e. after multiplying by some high-dimensional Euclidean space.

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Let $B$ be the smallest class of manifolds containing all compact manifolds with $\text{Ric} \geq 0$, and any manifold $E$ which is the total space of a smooth fiber bundle $F \to E \to B$ where $B \in B$, $F$ is a compact manifold of nonnegative Ricci curvature, and the structure group of the bundle lies in the isometry group of $F$. In other words, a manifold in $B$ is an iterated fiber bundle such that all the fibers, and the base at the very first step are compact manifolds of $\text{Ric} \geq 0$, and the structure groups lie in the isometry groups of the fibers. Note that $B$ is closed under products. Here is our main result.

**Theorem 1.1.** Let $B \in B$, and let $E(\xi)$ be the total space of a vector bundle $\xi$ over $B$. Then $E(\xi) \times \mathbb{R}^p$ admits a complete Riemannian metric of positive Ricci curvature for all sufficiently large $p$.

**Corollary 1.2.** Let $B$ be a compact manifold with $\text{Ric} \geq 0$. If $E(\xi)$ is the total space of a vector bundle $\xi$ over $B$, then $E(\xi) \times \mathbb{R}^p$ admits a complete Riemannian metric of positive Ricci curvature for all large $p$.

The case when $\xi$ is a trivial vector bundle in Theorem 1.1 generalizes the main result of [Wei88]. Actually, in Theorem 9.2 and Section 10 we prove a version of Theorem 1.1 for a larger class of base manifolds including some iterated fiber bundles with almost nonnegatively curved fibers. (A special case of this result with nilmanifolds as fibers was proved in [Wei89, Theorem 4]).

Theorem 1.1 should be compared with the results of J. Nash and L. Berard-Bergery [Nas79, BB78] who proved that the class of compact manifolds with $\text{Ric} > 0$ is closed under taking fiber bundles with structure groups lying in the isometry groups of the fibers, i.e. if $B, F$ admit $\text{Ric} > 0$, then so does $E$. Furthermore, any vector bundle of rank $\geq 2$ over a compact manifold with $\text{Ric} > 0$ carries a complete metric with $\text{Ric} > 0$ [Nas79, BB78]. Note that rank one vector bundles cannot carry $\text{Ric} > 0$ by the Cheeger-Gromoll splitting theorem.

By considering iterated bundles such that the fibers have large isometry groups (e.g. if the fibers are spheres, or compact Lie groups), one sees that $B$ contains many different topological types. For example, $B$ contains all nilmanifolds, or more generally, all iterated linear sphere bundles, as well as all iterated principal bundles (with compact fibers) over compact manifolds with $\text{Ric} \geq 0$.

All manifolds in $B$ admit metrics of almost nonnegative Ricci curvature (which was certainly known to Nash and Berard-Bergery), however, many of the manifolds in $B$ do not admit metrics with $\text{Ric} \geq 0$ (e.g. nilmanifolds). More examples are given in Section 11 where we show that if $E$ is a compact manifold with $\text{Ric}(E) \geq 0$ which fibers over a torus $T$, then the pullback of the bundle $E \to T$ to a finite cover of $T$ has a section.

Theorem 1.1 becomes false if in the definition of $B$ we do not assume that the structure group lies in the isometry group of the fiber. For example, Theorem 1.1 fails when $B$ is a compact Sol 3-manifold (which is a 2-torus bundle over a circle) because then $\pi_1(B)$ is not virtually nilpotent.

To prove 1.1 we show that any manifold in $B$ admits what we call a metric of *almost nonnegative Ricci curvature with good local basis* (see Section 7 for a precise definition), and
then we prove in Theorem 7.2 that the product of any manifold carrying such a metric with a high-dimensional Euclidean space admits a complete metric with $\text{Ric} > 0$.

As was suggested in [Wei89], it may well be true in general that if $E$ is a manifold of almost nonnegative Ricci curvature, then $E \times \mathbb{R}^p$ has a complete metric with $\text{Ric} > 0$ for large $p$. Our results is a further step in this direction.

The minimal value of $p$ coming from our construction generally depends on $\xi$, and typically is very large. It would be interesting to find obstructions for small $p$’s. For manifolds with infinite fundamental group there are obstructions due to M. Anderson [And90], e.g. no $\mathbb{R}^2$-bundle over a torus admits a complete metric with $\text{Ric} > 0$. By contrast, no obstructions are known in the simply-connected case, say it is unclear whether the product of $\mathbb{R}^2$ and a simply-connected compact Ricci-flat manifold can admit metrics with $\text{Ric} > 0$.

The structure of the paper is as follows. In sections 2–5 we review a well-known construction of metrics of almost nonnegative Ricci curvature on fiber bundles. We frequently refer to [Bes87, Chapter 9] for details. Sections 6–7 contain a curvature computation generalizing the main computation in [Wei88]. Section 8 provides an easy route to the proof of Corollary 1.2. Main technical results are proved in sections 9–10. In section 11 we give examples of iterated fiber bundles in the class $B$ which do not admit metrics of $\text{Ric} \geq 0$.

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2 Fiber bundles and Riemannian submersions

In sections 2–5 we let $\pi: E \to B$ be a Riemannian submersion of complete Riemannian manifolds with totally geodesic fibers. By [Bes87, 9.42], $\pi$ is a smooth fiber bundle whose structure group is a subgroup of the isometry group of the fiber; all fibers are isometric, and we denote a typical fiber by $F$ [Bes87, 9.56].

**Example 2.1.** Let $B$, $F$ be complete Riemannian manifolds where the metric on $F$ is invariant under a compact Lie group $G$. Let $\pi: E \to B$ be a smooth fiber bundle with fiber $F$ and structure group $G$. Then there exists a complete Riemannian metric on $E$, making $\pi$ a Riemannian submersion with totally geodesic fibers isometric to $F$ [Bes87, 9.59]. (To construct the metric, think of $E$ as an $F$-bundle associated with a principal $G$-bundle $P$ over $M$. Choose a connection on $P$ which defines a horizontal and vertical distributions on $E$. Introduce the metric on $E$ by making it equal to the metric on $B$, $F$ in the horizontal, vertical subspaces, respectively, and making the subspaces orthogonal). In particular, setting $F = G$ equipped with a left-invariant metric, we get a Riemannian submersion $P \to B$ where $P$ is any principal $G$-bundle $P$ over $M$.

**Example 2.2.** There is another useful metric on $E$. Namely, we think of $E$ as the quotient $(P \times F)/G$, take the product metric on $P \times F$ (where the metric on $P$ is defined as in Example 2.1), and give $E$ the Riemannian submersion metric (with horizontal spaces orthogonal to the fibers). The fibers in this metric are also totally geodesic [Nas79, 3.1] and isometric to the base of the Riemannian submersion $(G \times F) \to (G \times F)/G$ which is
diffeomorphic to $F$. For example, the total space of any rank $n$ vector bundle gets the Riemannian submersion metric from $(P \times \mathbb{R}^n)/O(n)$ where $\mathbb{R}^n$ is the Euclidean space; the fibers are isometric to $(O(n) \times \mathbb{R}^n)/O(n)$.

**Example 2.3.** One particularly simple kind of Riemannian submersion is a locally isometric flat bundle which can be described as follows. Start with complete Riemannian manifolds $B$, $F$, and a homomorphism $\rho: \pi_1(B) \to \text{Iso}(F)$, and consider the flat $F$-bundle $E$ over $B$ with holonomy $\rho$. If $\tilde{B} \times F$ is the universal cover of $B$, then the total space $E$ of the bundle is the quotient of $\tilde{B} \times F$ by the $\pi_1(B)$-action given by $\gamma(\tilde{b}, f) = (\gamma(\tilde{b}), \rho(\gamma)(f))$. The bundle projection $E \to B$ is induced by the projection $\tilde{B} \times F \to B$. The product metric on $\tilde{B} \times F$ defines a metric on $E$ which makes the projection $E \to B$ into a Riemannian submersion with totally geodesic fibers isometric to $F$, and zero $A$-tensor.

### 3 Choosing a basis

We use specific local trivializations of $\pi$ defined by taking a small strictly convex open ball $U$ in $B$ and considering horizontal lifts of the radial geodesics emanating from the center of $U$. Thus $\pi^{-1}(U)$ gets identified with $U \times F$ where the map $\{u\} \times F \to \{u\} \times F$ given by $(u, f) \to (u', f)$ is an isometry with respect to the induced metric on the fibers (see [Bes87, 9.56]).

To simplify the curvature computation we choose a basis on $E$ as follows. Fix an arbitrary point $e$ of $E$. Let $F_e$ be the fiber passing through $e$. At the tangent spaces $T_e F_e$, $T_{\pi(e)} B$, start with arbitrary orthonormal bases $\{\tilde{W}_i\}$, $\{\tilde{H}_j\}$, and extend $\tilde{W}_i$, $\tilde{H}_j$ to orthonormal vector fields also denoted $\tilde{W}_i$, $\tilde{H}_j$ on neighborhoods of $e \in F_e$, $\pi(e) \in B$, respectively, such that $\langle [\tilde{H}_i, \tilde{H}_j], \tilde{H}_k \rangle$ and $\langle [\tilde{W}_i, \tilde{W}_j], \tilde{W}_k \rangle$ vanish at $e$. (This can be achieved, for example, by choosing the extension so that $[\tilde{W}_i, \tilde{W}_j]$, $[\tilde{H}_i, \tilde{H}_j]$ vanish at $e$ or geodesic frame. The reason we care about the property can be seen in Lemma 6.2). Use the above local trivializations to extend these vector fields to vector fields $W_i$, $H_j$ defined on a neighborhood of $e$ in $E$. Thus, at any point of the neighborhood, $W_i$ is vertical, and $H_j$ is horizontal with $\pi_*(H_j) = \tilde{H}_j$. We conclude that $\{W_i, H_j\}$ is an orthonormal basis on a neighborhood of $e \in E$.

This basis has the property that $\langle [A, B], A \rangle|_e = 0$ for any $A, B \in \{W_i, H_j\}$. Indeed, at $e$ the following is true. First, since $\pi_*[H_i, H_j] = [\tilde{H}_i, \tilde{H}_j]$, we get $\langle [\tilde{H}_i, \tilde{H}_j], H_i \rangle = \langle [\tilde{H}_i, \tilde{H}_j], \tilde{H}_i \rangle = 0$. Similarly, $\pi_* W_j = 0$ implies $\langle [H_i, W_j], H_i \rangle = 0$. By construction $\langle [W_i, W_j], W_i \rangle = \langle [\tilde{W}_i, \tilde{W}_j], \tilde{W}_i \rangle = 0$. Finally, by the Koszul’s formula

$$\langle [W_i, H_j], W_i \rangle = -\langle \nabla_{\tilde{W}_i} W_i, H_j \rangle = 0$$

because the fibers are totally geodesic hence $\nabla_{\tilde{W}_i} W_i$ is vertical.

### 4 Submersions with bounded error term

We say that a Riemannian submersion with totally geodesic fibers has a **bounded error term** if there is an (independent of a point) constant $C > 0$ such that the tensors $\langle AU, AV \rangle$, $$
\((A_X, A_Y), \langle \delta A(X), U \rangle\), defined in [Bes87, 9.33], are bounded by \(C\) in absolute value, where \(U, V, X, Y\) are unit vector fields, \(U, V\) are vertical, and \(X, Y\) are horizontal. If \(E\) is compact, then by continuity \(\pi\) has a bounded error term, but there are some other examples. In fact, by [Bes87, 9.36], we have

\[
\langle AU, AV \rangle = \text{Ric}_E(U, V) - \text{Ric}_F(\hat{U}, \hat{V}),
\]

\[
2\langle A_X, A_Y \rangle = \text{Ric}_E(\hat{X}, \hat{Y}) - \text{Ric}_E(X, Y),
\]

\[
\langle \delta A(X), U \rangle = -\text{Ric}_E(X, U),
\]

so if \(\text{Ric}_B, \text{Ric}_F, \text{Ric}_E\) are bounded in absolute value, then \(\pi\) has a bounded error term (Here \(\hat{X}\) denotes \(\pi_* X\), and \(\hat{U}\) denotes the restriction of \(U\) to the fiber, etc).

**Example 4.1.** The projection of any rank \(n\) vector bundle \(E \to B\) over a compact manifold \(B\) can be made into a Riemannian submersion as in Example 2.2. Here \(E\) gets the metric as the base of the Riemannian submersion \(p: (P \times \mathbb{R}^n) \to (P \times \mathbb{R}^n)/O(n)\). By a simple computation (done e.g. in [And87, Page 361]), the \(A\)-tensor of \(p\) is such that \(|A_X Y|\) is uniformly bounded above for any \(p\)-horizontal orthogonal unit vector fields \(X, Y\). Then since the sectional curvature of \(P \times \mathbb{R}^n\) is bounded above and below, so is the sectional curvature of \(E\) by the O’Neill’s formula. By the same argument the sectional curvature of \((O(n) \times \mathbb{R}^n)/O(n),\) which is the fiber of \(E \to B,\) is bounded above and below, hence the submersion \(E \to B\) has a bounded error term.

## 5 Scaling Riemannian submersions

Given \(t \in (0, 1]\), let \(g_{t,E}\) be the family of metrics on \(E\) defined by

\[
g_{t,E}(U + X, V + Y) = t^2 g_E(U, V) + g_E(X, Y),
\]

where \(U, V\) are vertical, and \(X, Y\) are horizontal. We refer to [Bes87, 9G] for more information on such metrics (note that Besse scales the metric on fibers by \(t\) while we use \(t^2\)). By [Bes87, 9.70] (note the wrong sign in 9.70c), applied to the submersion \(\pi: E \to B,\) we get

\[
\text{Ric}_{g_{t,E}}(U, V) = \text{Ric}_{g_{t,F}}(\hat{U}, \hat{V}) + t^4 \langle AU, AV \rangle,
\]

\[
\text{Ric}_{g_{t,E}}(X, Y) = \text{Ric}_{g_{B}}(\hat{X}, \hat{Y}) - 2t^2 \langle A_X, A_Y \rangle,
\]

\[
\text{Ric}_{g_{t,E}}(X, U) = -t^2 \langle \delta A(X), U \rangle.
\]

Here \(g_{t,F}\) is the metric on \(F\) induced by \(g_{t,E}\), in particular, \(\text{Ric}_{g_{t,F}} = t^{-2} \text{Ric}_F\).

For example, if in the construction of \(\{H_i, W_j\}\), we assume that \(\{\hat{W}_i\}, \{\hat{H}_j\}\) are orthonormal bases of eigenvectors diagonalizing the Ricci tensors \(\text{Ric}_{g_{F}}, \text{Ric}_B\) (so that for \(i \neq j, \text{Ric}_F(\hat{W}_i, W_j) = 0 = \text{Ric}_B(\hat{H}_i, \hat{H}_j)\)), and if \(\pi\) has a bounded error term, then

\[
|\text{Ric}_{g_{t,E}}(A, B)| \leq Ct \text{ for any distinct } A, B \in \{H_i, W_j\},
\]

\[
\text{Ric}_{g_{t,E}}(\frac{W_i}{\tau}, \frac{W_i}{\tau}) \geq \text{Ric}_F(\hat{W}_i, \hat{W}_i) t^{-2},
\]

\[
\text{Ric}_{g_{t,E}}(H_i, H_i) \geq \text{Ric}_B(\hat{H}_i, \hat{H}_i) - Ct^2.
\]

So if \(\text{Ric}(F) \geq 0\) and \(\text{Ric}(B) \geq 0\), then \(\text{Ric}_{g_{t,E}}(W_i, W_j) \geq 0\) and \(\text{Ric}_{g_{t,E}}(H_i, H_i) \geq -Ct^2\).
6 Computing the Ricci curvature of a warped product

In this section we present a generalization of the main computation of [Wei88]. Let $E$ be an $n$-manifold with a smooth family of complete Riemannian metrics $g_r$, $r \geq 0$. Let $f$ be a function of $r$ to be specified later. Consider a metric on $N = E \times S^{p-1} \times (0, \infty)$ given by $g = s_r + dr^2$ where $s_r = g_r + f^2 ds^2$ and $ds^2$, $dr^2$ are the canonical metrics on $S^{p-1}$, and $(0, \infty)$.

Note that $N$ is an open subset in a manifold diffeomorphic to $E \times \mathbb{R}^p$. Our goal is to find the conditions ensuring that $g$ extends to a complete metric on $E \times \mathbb{R}^p$ with $Ric(g) > 0$ for all sufficiently large $p$.

For each point $e \in E$ assume that there is a basis of vector fields $\{X_i\}$ on a neighborhood $U_e \subset E$ of $e$ such that for any $r \geq 0$, $g_r(X_i, X_j) = 0$ on $U_e$ if $i \neq j$. Fix one such a basis for each $e$. Let $h_i(r) = \sqrt{g_r(X_i, X_i)}$ so that $Y_i = X_i/h_i$ form a $g_r$-orthonormal basis on $U_e$ for any $r \geq 0$. Since $X_i \neq 0$ and $g_r$ is nondegenerate, $h_i$ is a positive function on $[0, \infty)$. Assume furthermore that $h_i$ is smooth and $h_i^{(odd)}(0) = 0$ (i.e. all odd derivatives of $h_i$ at zero vanish). An argument similar to [Pet98, Page 13] shows that $g$ is a smooth complete Riemannian metric on $E \times \mathbb{R}^p$ if $f(0) = 0$, $f(r) > 0$ for $r > 0$, $f'(0) = 1$, $f^{(even)}(0) = 0$.

For the rest of the section we fix an arbitrary point $(e, s, r) \in N$. Our goal is to compute the Ricci tensor of $g$ at the point in terms of $f$, $h_i$'s, and the Ricci tensor of $g_r$.

At a neighborhood of $s \in (S^{p-1}, ds^2)$ choose an orthonormal frame $\{V_j\}$ with $\nabla ds^2 V_k|s = 0$ for any $j, k$ where $\nabla ds^2$ is the Riemannian connection for $ds^2$. In particular, $[V_i, V_j]|s = 0$. Denote $U_j = V_j/f$ and $\partial_r = \partial/\partial r$ and so that

$$\beta_r = \{\partial_r, U_1, \ldots U_{p-1}, Y_1, \ldots, Y_n\}, \quad \beta = \{U_1, \ldots U_{p-1}, Y_1, \ldots, Y_n\}$$

are orthonormal frames at $(e, s)$ and $(e, s)$, respectively.

We start with a few elementary observations which we use throughout this section. Any function of $r$ has zero derivative in the direction of $X_i$ (because $X_i$ is independent of $r$), or $V_i$ (since $V_i$ is tangent to the level surface of the function). Hence $[Y_i, Y_j] = \frac{1}{h_i h_j} [X_i, X_j]$ and $[U_i, U_j]|s = \frac{1}{f^2} [V_i, V_j]|s = 0$. Also the flow of $\partial_r$ preserves $X_i$, $V_j$ so $[\partial_r, X_i] = 0 = [\partial_r, V_j]$ which implies $[Y_i, \partial_r] = \frac{h_i'}{h_i} Y_i$ and $[U_j, \partial_r] = \frac{f'}{f} U_j$.

We frequently use the Koszul's formula for the Riemannian connection of $g$ computed in the orthonormal basis, namely for any $A, B, C \in \beta$

$$2 \langle \nabla_A B, C \rangle_g = \langle [A, B], C \rangle_g + \langle [C, A], B \rangle_g + \langle [C, B], A \rangle_g.$$

In particular, the Koszul's formula implies $\nabla_Y \partial_r = \frac{h_i'}{h_i} Y_i$ and $\nabla_{U_j} \partial_r = \frac{f'}{f} U_j$ so that $\nabla_{\partial_r} Y_i$, $\nabla_{\partial_r} U_j$ vanish. Also $\nabla_{\partial_r} \partial_r = 0$. For any $i, j$, $[U_i, U_j], [Y_i, U_j], [Y_i, Y_j]$ are all tangent to $E \times S^{p-1}$ so they all have zero $\partial_r$-component. Therefore, by the Koszul's formula the terms $\langle \nabla_Y Y_j, \partial_r \rangle$, $\langle \nabla_Y U_j, \partial_r \rangle$, $\langle \nabla_{U_j} U_j, \partial_r \rangle$ all vanish for $i \neq j$. The Koszul's formula also implies $\langle \nabla_Y Y_i, \partial_r \rangle = \langle \partial_r, Y_i \rangle Y_i = -\frac{h_i'}{h_i} Y_i$ and similarly $\langle \nabla_{U_i} U_i, \partial_r \rangle = -\frac{f'}{f} U_i$.

The second fundamental form of the submanifold $E \times S^{p-1} \subset N$ is given by $II(A, B) = \langle \nabla_A B, \partial_r \rangle \partial_r$ for $A, B \in \beta$. Thus, $II(Y_i, Y_i) = -\frac{h_i'}{h_i} \partial_r$, $II(U_i, U_i) = -\frac{f'}{f} \partial_r$ while all the mixed
terms $II(Y_i, U_j)$ as well as the terms $II(U_i, U_j)$, $II(Y_i, Y_j)$ for $i \neq j$ vanish. Now the Gauss equation [KN63, VII.4.1] gives the sectional curvature of $N$ in terms of the sectional curvature of $(E \times S^{p-1}, s_r)$ as follows:

$$K_g(Y_i, U_j) = K_{s_r}(Y_i, U_j) - \frac{h'_i f'}{h_i f}$$

$$K_g(Y_i, Y_j) = K_{s_r}(Y_i, Y_j) - \frac{h'_i h'_j}{h_i h_j}$$

$$K_g(U_i, U_j) = K_{s_r}(U_i, U_j) - \left(\frac{f'}{f}\right)^2.$$  

Since $s_r$ is the product metric on $E \times S^{p-1}$, we get $\nabla_{U_i} Y_j = 0 = \nabla_{Y_j} U_i$ so that the curvature $(2,2)$-tensor $R_{s_r}(U_i, Y_j)$ vanishes. Hence $K_{s_r}(U_i, U_j) = 0$ and $\langle R_{s_r}(A, B)C, D \rangle = 0$ for any $A, B, C, D \in \beta$ unless $A, B, C, D$ are all tangent to $E$ or $S^{p-1}$. Also

$$\langle R_{s_r}(Y_i, Y_k)Y_i, Y_j \rangle = \langle R_{g_r}(Y_i, Y_k)Y_i, Y_j \rangle, \quad \langle R_{s_r}(U_i, U_k)U_i, U_j \rangle = \langle R_{f^2 ds^2}(U_i, U_k)U_i, U_j \rangle,$$

$$K_{s_r}(Y_i, Y_j) = K_{g_r}(Y_i, Y_j), \quad K_{s_r}(U_i, U_j) = K_{f^2 ds^2}(U_i, U_j) = 1/f^2.$$

Now we turn to the terms of the curvature tensor involving $\partial_r$ and the mixed terms.

**Lemma 6.1.**

1. $R_g(\partial_r, U_i)\partial_r = \left(\frac{f''}{f} \right) U_i$ and $R_{g_r}(\partial_r, Y_i)\partial_r = \left(\frac{h''}{h} \right) Y_i$.

2. $K_g(U_i, \partial_r) = -\left(\frac{f''}{f} \right)$ and $K_g(Y_i, \partial_r) = -\left(\frac{h''}{h} \right)$.

3. $Ric_g(Y_i, Y_j) = Ric_{g_r}(Y_i, Y_j)$ and $Ric_g(U_i, U_j) = 0$ for $i \neq j$.

4. $Ric_g(U_i, Y_j) = 0$.

**Proof.** (1) follows directly from $\nabla_{\partial_r} \partial_r = 0$, $\nabla Y_i \partial_r = \frac{h'_i}{h} Y_i$, $\nabla U_j \partial_r = \frac{f'}{f} U_j$, $\lbrack Y_i, \partial_r \rbrack = \frac{h'_i}{h} Y_i$, and $\lbrack U_j, \partial_r \rbrack = \frac{f'}{f} U_j$.

(2) follows from (1).

(3)-(4) By the Gauss equation [KN63, VII.4.1], we have $R_g(A, C, C, B) = R_{s_r}(A, C, C, B)$ for any $A, B, C \in \beta, A \neq B$. By part (1), $R_g(A, \partial_r)\partial_r$ is proportional to $A$, so $R_g(A, \partial_r, \partial_r, B) = 0$ if $A \neq B$. Now $Ric_g(A, B)$ is the sum of $R_g(A, \partial_r, \partial_r, B)$ and all the terms $R_g(A, C, C, B)$ so we get $Ric_g(A, B) = Ric_{s_r}(A, B)$. Using the above formulas for $R_{s_r}$, with $i \neq j$, we get $Ric_{s_r}(Y_i, Y_j) = Ric_{g_r}(Y_i, Y_j)$, and $Ric_{s_r}(U_i, U_j) = Ric_{f^2 ds^2}(U_i, U_j) = 0$ where the last equality is true since $f^2 ds^2$ has constant sectional curvature. Finally, $Ric_{s_r}(Y_i, U_j) = 0$ since all the involved curvature tensor terms vanish. 

**Lemma 6.2.**

1. $\langle R_g(\partial_r, Y_i)Y_i, Y_j \rangle = \langle Y_i, Y_j \rangle Y_i \left(\frac{h'_i}{h} + \frac{h'_j}{h_j} \right)$.

2. $\langle R_g(\partial_r, A)B, C \rangle = 0$ for any $A, B, C \in \beta$ unless $A, B, C$ are all tangent to $E$.

3. $Ric_g(\partial_r, U_i) = 0$.

4. $Ric_g(\partial_r, Y_j) = \sum_{i \neq j} \langle Y_i, Y_j \rangle Y_i \left(\frac{h'_i}{h_i} + \frac{h'_j}{h_j} \right)$.

**Proof.** Using the Koszul's formula, we compute the components of $\nabla_{Y_i} Y_i$ in the basis $\beta_r$ to deduce that $\nabla_{Y_i} Y_i = \sum_{j} \langle Y_j, Y_i \rangle Y_j - \frac{h'_i}{h_i} \partial_r$. Then using that $\nabla_{\partial_r} Y_i = 0$, $\lbrack Y_i, \partial_r \rbrack = \frac{h'_i}{h} Y_i$, we can compute the curvature tensor $R_g(\partial_r, Y_i)Y_i$, and then its components in $\beta$ so that (1)-(2) follow by a straightforward computation. A similar argument gives the result for $U_i$’s the only difference being $\lbrack U_i, U_j \rbrack = 0$ at $(e, s, r)$. Finally, (3)-(4) follow from (1)-(2) by substitution.
In summary, we have

\[ \text{Ric}_g(U_i, U_i) = (p - 2) \frac{1 - (f')^2}{f^2} - \frac{f'}{f} \sum_{i=1}^{n} \frac{h'_i}{h_i} - \frac{f''}{f} \]  

(6.5)

\[ \text{Ric}_g(Y_i, Y_i) = \text{Ric}_{g_t}(Y_i, Y_i) - (p - 1) \frac{f'h'_i}{fh_i} - \frac{h'_i}{h_i} \sum_{k \neq i} \frac{h'_k}{h_k} - \frac{h''_i}{h_i} \]  

(6.6)

\[ \text{Ric}_g(\partial_r, \partial_r) = -(p - 1) \frac{f''}{f} - \sum_{i=1}^{n} \frac{h''_i}{h_i} \]  

(6.7)

\[ \text{Ric}_g(U_i, U_j) = 0 \text{ for } i \neq j, \quad \text{Ric}_g(\partial_r, U_i) = \text{Ric}_g(U_i, Y_j) = 0 \text{ for all } i, j, \]  

(6.8)

\[ \text{Ric}_g(\partial_r, Y_j) = \sum_{i \neq j} \langle [Y_i, Y_j], Y_i \rangle \left( \frac{h'_i}{h_i} + \frac{h'_j}{h_j} \right), \]  

(6.9)

\[ \text{Ric}_g(Y_i, Y_j) = \text{Ric}_{g_t}(Y_i, Y_j) \text{ for } i \neq j. \]  

(6.10)

7 Turning almost nonnegative into positive

For \( q > 0 \) and \( c, m \geq 0 \), define the class \( \mathcal{M}_q(c, m) \) of smooth manifolds by requiring that any \( E \in \mathcal{M}_q(c, m) \) has a smooth family of complete Riemannian metric \( g_t, t \in (0, 1] \) such that any \( e \in E \) has a neighborhood \( U \) with a basis of vector fields \( X_i \) such that

1. \( Y_i = X_i/t^m_i \) form a \( g_t \)-orthonormal basis on \( U \) for some \( m_i \in [0, m] \), depending on \( e \),
2. at the point \( e \) the following holds: \( \langle [Y_i, Y_j], Y_i \rangle g_t = 0, \text{Ric}_{g_t}(Y_i, Y_i) \geq -ct^q \), for all \( i, j \), and \( |\text{Ric}_{g_t}(Y_i, Y_j)| \leq ct^q \) for \( i \neq j \).

Let \( \mathcal{M}_q = \cup_{c, m} \mathcal{M}_q(c, m) \) and \( \mathcal{M} = \cup_q \mathcal{M}_q \). We refer to \( \mathcal{M} \) as the class of almost nonnegatively Ricci curved manifolds with good local basis. We show in Theorem 7.2 that if \( E \in \mathcal{M} \), then \( E \times \mathbb{R}^p \) admits a complete metric with \( \text{Ric} > 0 \) for all large \( p \).

Reparametrizing \( t \) by \( t' \) for \( r > 0 \), we get \( \mathcal{M}_{rq}(c, m) = \mathcal{M}_q(c, rm) \). Also rescaling \( g_t \) by \( t' \), we have \( \mathcal{M}_{q-2r}(c, m) = \mathcal{M}_q(c, r+m) \) for \( 0 < r < q/2 \). Note that \( \mathcal{M}_q(c, m) \subset \mathcal{M}_s(c, m) \) if \( q \geq s \).

**Example 7.1.**

1. If \( E \) has a complete metric \( g \) with \( \text{Ric} \geq 0 \), then \( E \in \mathcal{M}_q(0, 0) \) for every \( q \). (Take \( g_t = g \) for all \( t \), choose a \( g \)-orthonormal basis \( \{X_i\} \) of \( T_eE \) diagonalizing the Ricci tensor at \( e \), and extend it to an orthonormal basis of vector fields on \( U \) satisfying \( [X_i, X_j] = 0 \).)
2. According to [Wei89], any \( n \)-dimensional nilmanifold, (or even any \( n \)-dimensional infranilmanifold [Wei89, Theorem 3]) lies in \( \mathcal{M}_q(c, 2^{n-2}(q-1) + 1) \) for every \( q \), where \( c \) depends on the structure constants and \( n \).

Using the formulas of Section 6, we now prove the following.

**Theorem 7.2.** For each \( E \in \mathcal{M} \) there is a complete metric on \( E \times \mathbb{R}^p \) with \( \text{Ric} > 0 \) for all \( p \) large.
If $E \in \mathcal{M}$, then $E \in \mathcal{M}_q$ for some $q > 0$. By reparametrizing and rescaling we can assume that $E \in \mathcal{M}_2$ with all $m_i$ positive. Now Theorem 7.2 follows from the theorem below.

**Theorem 7.3.** If $E \in \mathcal{M}_2(c,m)$ is an $n$-manifold with all $m_i$ positive (equivalently, $E \in \mathcal{M}_q$, where $q > 2$), then there is an explicit function $k(n,c,m)$ such that for all $p > k(n,c,m)$, there is a complete Riemannian metric $g$ on $E \times \mathbb{R}^p$ of positive Ricci curvature.

**Proof.** Since $E \in \mathcal{M}_2(c,m)$ with all $m_i$ positive, by parametrizing $t = h(r)$ where $h(r) = (1 + r^2)^{-1}$, $r > 0$, we get a smooth family of complete Riemannian metrics $g_r$, $r > 0$, such that at a neighborhood $U$ of any point of $e \in E$ there is a basis of vector fields $\{X_i\}$ such that $Y_i = X_i/h^{m_i}$ form a $g_r$-orthonormal basis on $U$ where $m_i \in (0, m]$, and at $e$ we have $\langle [Y_i, Y_j], Y_i \rangle_{g_r} = 0$ for all $i, j$, and $|\text{Ric}_{g_r}(Y_i, Y_j)| \leq ch^2$ if $i \neq j$, and $\text{Ric}_{g_r}(Y_i, Y_i) \geq -ch^2$ for all $i$.

Let $g = g_r + dr^2 + f^2 ds^2$ be the metric on $E \times \mathbb{R}^p$, where $ds^2$ are the canonical metrics on $S^{p-1}$ and $f(r) = r(1 + r^2)^{-1/4}$. Now using the Ricci curvature formulas (6.5)-(6.10), we conclude that all the mixed terms vanish except $\text{Ric}_g(Y_i, Y_j)$, and $|\text{Ric}_g(Y_i, Y_j)| \leq ch^2$ when $i \neq j$.

Furthermore, by a straightforward computation

$$
\text{Ric}_g(Y_i, Y_i) \geq h^2 \left( r^2(pK - L) + pR - S \right),
$$

$$
\text{Ric}_g(\partial_r, \partial_r) = h^2 \left( r^2(pK - L) + pR - S \right),
$$

$$
\text{Ric}_g(U_i, U_i) = (p-2)f^{-2}(1 - (f')^2) + h^2(Kr^2 + R),
$$

where $K, L, R, S$ are some explicit linear functions of $n, c, m$ depending on a particular element of $\beta$, (may be different in different equations above), and $K, R$ are positive. Note that $f^{-2}(1 - (f')^2) \geq h^2(\frac{3}{4} + r^2)$. Therefore, the matrix of the Ricci tensor in the basis $\beta_r$ is positive definite for all sufficiently large $p > k(n,c,m)$, where $k(n,c,m)$ depends on $n,c,m$ and is independent of $r$. 

\qed

### 8 Metrics on fiber bundles

This section provides an easy route to the proof of Corollary 1.2.

**Theorem 8.1.** Let $\pi: E \to B$ be a Riemannian submersion with totally geodesic fibers isometric to $F$, and a bounded error term. Assume that $E$ is complete, and $B, F$ have nonnegative Ricci curvature. Then $E \times \mathbb{R}^p$ admits a complete Riemannian metric of positive Ricci curvature for all sufficiently large $p$.

**Proof.** Write $TE$ as the sum $V \oplus H$ of a vertical and a horizontal subbundles. Fix a point $e \in E$. By Section 3, we can find an orthonormal basis $\{X_k\}$ of vector fields on a neighborhood of $e \in E$ with the following properties:

1. each $X_k$ is either horizontal or vertical,
2. if $X_k \in H$, then $X_k$ is a horizontal lift of a vector field $X_k$ on $B$,
3. $\text{Ric}_B(X_k, X_j)|_e = 0$ for any $X_k, X_j \in H$ with $k \neq j$,
(4) $\text{Ric}_F(\tilde{X}_k, \tilde{X}_j)|_e = 0$ for any $X_k, X_j \in V$ with $k \neq j$,
(5) $([X_k, X_j], X_k)|_e = 0$ for all $k, j$.

For $t \in (0, 1]$, define a metric on $E$ by $g_{t,E}$ where $g_{t,E}(X+V, Y+U) = g_E(X, Y) + t^2 g_E(U, V)$ where $g_E$ is the original metric on $E$, $X, Y \in \mathcal{H}$, and $U, V \in V$. Setting $Y_k = X_k$ if $X_k \in \mathcal{H}$, and $Y_k = X_k/t$ if $X_k \in V$, we get a $g_{t,E}$-orthonormal basis $\{Y_k\}$ on a neighborhood of $e$.

If $C > 0$ bounds the error term of $\pi$, then the Ricci tensor $\text{Ric}_{g_{t,E}}$ computed in the basis $\{Y_k\}$ satisfies $|\text{Ric}_{g_{t,E}}(Y_k, Y_j)| \leq Ct$ if $k \neq j$, and $\text{Ric}_{g_{t,E}}(Y_k, Y_k) \geq -Ct^2 \geq -Ct$ for each $k$ (where we used $\text{Ric}_B \geq 0$, $\text{Ric}_F \geq 0$ for the latter inequality). Now we are in position to apply Theorem 7.2 which completes the proof.

**Corollary 8.2.** Let $B, F$ be compact Riemannian manifolds of nonnegative Ricci curvature, and let $E \to B$ be a smooth fiber bundle with fiber $F$ and structure group in the isometry group of $F$. Then $E \times \mathbb{R}^p$ admits a complete Riemannian metric of positive Ricci curvature for all sufficiently large $p$.

**Proof.** By Example 2.1 $E$ has a metric making the projection $E \to B$ a Riemannian submersion with totally geodesic fibers isometric to $F$. Since $E$ is compact, the submersion has a bounded error term, so we are done by Theorem 8.1.

**Proof of Corollary 1.2.** By Example 4.1 there is a complete Riemannian metric on $E$ making the bundle projection $E \to B$ a Riemannian submersion with totally geodesic fibers and bounded error term so we are done by Theorem 8.1.

**Remark 8.3.** It would be interesting to get a realistic lower bound on $p$. The bound we get is an explicit function of $C$ and $\dim(B)$, $\dim(F)$ which has little practical value since we do not know how to estimate $C$. In the case (covered by Theorem 1.1) when $B$ is a nilmanifold the constant $C$ can be estimated in terms of $\dim(B)$, and the structure constants of the simply-connected nilpotent group covering $B$. This indicates that $C$ cannot be chosen independently of the vector bundle.

9 Metrics on iterated fiber bundles I

In this section we prove a technical lemma which implies Theorem 1.1 (in fact, in Theorem 9.2 we deal with a larger class of base spaces). To prove Theorem 1.1 we show that any manifold in $\mathcal{B}$ belongs to the class $\mathcal{M}$ of manifolds of almost nonnegative Ricci curvature with good local basis, so that Theorem 7.2 applies.

Let $G$ be a Lie group, $B$ be a smooth manifold, and $P \to B$ be a principal $G$-bundle over $B$ with a connection. Let $\pi: E \to B$ be a smooth fiber bundle with fiber $F$ associated with $P$. The connection on $P$ defines a decomposition of the tangent bundle to $E$ as the sum of a vertical and a horizontal subbundles, $V$ and $H$. Let $(E, g^E_t)$, $(B, g^B_t)$ be smooth families of complete Riemannian metrics parametrized by $t \in (0, 1]$, and such that for each $t$, $\pi$ is a Riemannian submersion with totally geodesic fibers and horizontal space $H$. We denote the induced metric on $F$ by $g^F_t$. 

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Lemma 9.1. Let \( \pi: (E, g_t^E) \rightarrow (B, g_t^B) \) be a family of Riemannian submersions as above. Assume that \((B, g_t^B) \in \mathcal{M}_q(c_B, m_B), (F, g_t^F) \in \mathcal{M}_r(c_F, m_F)\), and there are nonnegative constants \(L_a, L_b, L_f, b, f\) such that

1. \(|A_X Y|_{g_t^E} \leq L_a\) for \(t = 1\) and all pairs of unit orthogonal vector fields \(X, Y \in H\),
2. \(|K_{g_t^E}| \leq L_b/t^b\) and \(|K_{g_t^F}| \leq L_f/t^f\).

Then the following holds for some positive constants \(Q_4, Q_5\) and \(m = \max\{b, 2m_B, f\}\).

(A) If \(r \geq 2m + 3q\), then \((E, g_t^E) \in \mathcal{M}_q(c_E, m_E)\) for some \(g_t^E\) satisfying \(|K_{g_t^E}| \leq Q_4/t^{2m + 3q + f}\), where \(m_E = \max\{m_B, m_F\}\).

(B) If \(L_f = 0 = b\), then for each \(p\), \((E, g_t^{E,p}) \in \mathcal{M}_p\) for some \(g_t^{E,p}\) satisfying \(|K_{g_t^{E,p}}| \leq Q_5\).

(C) If the \(A\)-tensor of \(\pi\) is everywhere zero (i.e. \(\pi\) is a locally isometric flat bundle), then for each \(p\), \((E, g_t^{E,p}) \in \mathcal{M}_p\) for some \(g_t^{E,p}\) satisfying \(|K_{g_t^{E,p}}| \leq L_t\cdot t^{-p/k} + L_ft^{-p/f}\) where \(k = \min\{r, q\}\).

Proof. For \(s \in (0, 1]\), define a metric on \(E\) by \(g_{t,s}^E(X + U, Y + V) = g_t^E(X, Y) + s^2g_t^E(U, V)\) where \(X, Y \in H\), and \(U, V \in V\). Fix a point \(e \in E\), and follow the procedure in section 3 to construct an orthonormal basis \(\{Y_i\}\) at a neighborhood of \(e\) by combining the bases at \(\pi(e) \in B\), and \(e \in F\). In particular, \((Y_i, Y_j)\) \(g_{t,s}\) vanishes at \(e\).

First, we estimate the \(A\)-tensor of \(\pi\). Since \(A_X Y = [X, Y]V\) for \(X, Y \in H\), and the subbundles \(H, V\) are independent of \(t\), so is \(A_X Y\). For \(Y_i, Y_j \in H\) we get:

\(|A_{Y_i Y_j}|_{g_t^E} = t^{-m_i - m_j}|A_{X_i X_j}|_{g_t^E} \leq t^{-m_i - m_j}|A_{X_i X_j}|_{g_t^E} \leq L_a t^{-m_i - m_j} \leq L_a t^{-2m_B},\)

where the first inequality uses that the \(g_t\)-length of any vector at \(e\) is bounded above by its \(g_{t,s}\)-length, since \(t \leq 1\). Thus, \(|A_X Y|_{g_t^E} \leq (\dim B)^2 L_a t^{-2m_B}\), for any unit vector fields \(X, Y \in H\). Now let \(U \in V\). Since \(A_X U\) is horizontal, and \(<A_X U, Y_j>_{g_t^E} = -<A_X Y_j, U>_{g_t^E}\), we deduce by computing in the basis \(\{Y_j\}\) that \(|A_X U|_{g_t^E} \leq (\dim B)^2 L_a t^{-2m_B}\).

From the O’Neill formulas for the sectional curvature [Bes87, 9.29] of \(g_{t,s}^E\) we now see that

\(|K_{g_{t,s}^E}| = \frac{4(\dim B)^2 L_a s^2}{t^{2m_B}} + \frac{|K_{g_t^F}|}{s^2} \leq \frac{L_b}{t^b} + \frac{4(\dim B)^2 L_a s^2}{t^{2m_B}} + \frac{L_f}{s^2 t^f} = L(t, s).\)

Each component of the Ricci tensor \(\text{Ric}_{g_{t,s}^E}(Y_i, Y_j)\) is the sum of the components of the curvature tensor, which in turn is the sum of sectional curvatures. Therefore, we get the estimate: \(|\text{Ric}_{g_{t,s}^E}(Y_i, Y_j)| \leq Q_1 \cdot L(t, s)\) for any \(i, j\) and some constant \(Q_1 > 0\). Setting \(s = 1\), and using \(|\text{Ric}_{g_t^B}| \leq (\dim B)L_b/t^b\) and \(|\text{Ric}_{g_t^F}| \leq (\dim F)L_f/t^f\), we conclude that the error of the Riemannian submersion \(\pi: (E, g_{t,1}^E) \rightarrow (B, g_{t,1}^B)\) is bounded above by \(Q_2 \cdot L(t, 1)\) for some constant \(Q_2 > 0\).

To prove (A), let \(m = \max\{b, 2m_B, f\}\) so that \(Q_2 \cdot L(t, 1) \leq Q_3/t^m\) for a constant \(Q_3 > 0\). Setting \(g_t^E = g_{t,s}^E\), where \(s = t^{m+q}\), we get from (5.2)-(5.4) that \(|\text{Ric}_{g_t^E}(Y_i, Y_j)| \leq Q_3 t^q\) for any \(i \neq j\), \(\text{Ric}_{g_t^E}(Y_i, Y_i) \leq -(c_B + Q_3) t^q\) for any \(Y_i \in H\), and for any \(Y_i \in V\),

\(|\text{Ric}(Y_i, Y_i)_{g_t^E} \geq -c_F t^q t^{-2m-2q} \geq -c_F t^q,\)

where the last inequality follows from \(r \geq 2m + 3q\). Thus, \((E, g_t^E) \in \mathcal{M}_q\). Note that \(|K_{g_t^E}| \leq L(t, t^{m+q}) \leq Q_4/t^{2m+2q+f}\) for some \(Q_4 > 0\).
To prove (B) note that $b = 0 = L_f$ implies that $L(t, s) = L_b + 4(\dim B)^2L_a s^2/t^{2m_B}$. Let $\bar{g}_E^t = g_{E,s}^t$ with $s = t^{2m_B+1}$ so $|K_{\bar{g}_E}| \leq Q_5$ for a constant $Q_5 > 0$. Now the error of $\pi$: $(E, g_{E,1}^t) \to (B, g_{B,1}^t)$ is bounded by $Q_6 t^{-2m_B}$ for a constant $Q_6 > 0$, so $|\text{Ric}_{\bar{g}_E}(Y_i, Y_j)| \leq Q_6 t$ if $i \neq j$, $|\text{Ric}_{\bar{g}_E}(Y_i, Y_i)| \geq -Q_6t - c_B t^q \geq -t^k(Q_6 + c_B)$ where $k = \min\{1, q\} > 0$. Thus, $(E, \bar{g}_E^t) \in \mathcal{M}_k$ and so reparametrizing $\bar{g}_t^{E,p} = \bar{g}_{\bar{g}_{E,p}}^t$, we have $(E, \bar{g}_t^{E,p}) \in \mathcal{M}_p$ for any $p > 0$ and $|K_{\bar{g}_t^{E,p}}| \leq Q_5$.

To prove (C) note that the metric $g_{E,1}^t$ satisfies $|K_{g_{E,1}^t}| \leq L_b t^{-b} + L_1 t^{-f}$ because $L_a = 0$, and $(E, \bar{g}_t^{E}) \in \mathcal{M}_k$ where $k = \min\{r, q\} > 0$. Reparametrizing $\bar{g}_t^{E,p} = \bar{g}_{\bar{g}_{E,p}}^t$, we get the desired metric.

**Theorem 9.2.** Let $B$ be a compact manifold with a family of metrics $g_t^B$ satisfying $(B, g_t^B) \in \mathcal{M}_q$ and $|K_{g_t^B}| \leq L_b/t^b$ for some $q > 0$, $L_b, b \geq 0$. If $\mathbb{R}^n \to E \to B$ is a vector bundle over $B$, then $E \in \mathcal{M}_q$.

**Proof.** Fix $q > 0$. By Example 4.1 we make the projection $E \to B$ into a Riemannian submersion. Namely, consider the principal $O(n)$-bundle $P \to B$ such that $E \to B$ is associated with $P$, fix a connection on $P$, and define a family of metrics on $P$ as in Example 2.1; in particular, the fiber is $O(n)$ with an independent of $t$ biinvariant metric and $P \to B$ is a Riemannian submersion with totally geodesic fibers for each $t$. Then $E$ gets the metric as the base of the Riemannian submersion $(P \times \mathbb{R}^n) \to (P \times \mathbb{R}^n)/O(n)$. Now the projection $E \to B$ is a Riemannian submersion with totally geodesic fibers isometric to $(O(n) \times \mathbb{R}^n)/O(n)$. The metric on the fibers is independent of $t$, and has sectional curvatures within $[0, L_f]$ for some $L_f > 0$. Note that the horizontal and vertical spaces on $P$ are independent of $t$, hence $E$ enjoys the same property. As we mentioned in Example 4.1, the $A$-tensor of $E \to B$ satisfies Lemma 9.1(1). It remains to apply Lemma 9.1(A).

**Proof of Theorem 1.1.** By Theorems 7.2, 9.2, it suffices to find a family of metrics $g_t^B$ on $B$ satisfying $(B, g_t^B) \in \mathcal{M}_q$ and $|K_{g_t^B}| \leq L_b/t^b$. We argue inductively from the definition of the class $\mathcal{M}$. Given metrics on $F, B$, we always equip $E$ with the metric as in Example 2.1. If $B$ is compact with $\text{Ric} \geq 0$, then $B \in \mathcal{M}_r$ for any $r$ by Example 7.1, and the sectional curvature of $B$ is bounded by a constant. The induction step follows from Lemma 9.1(A), where the assumption (1) holds by compactness.

## 10 Metrics on iterated fiber bundles II

In this section we give more sophisticated examples of base manifolds for which Theorem 9.2 applies. Roughly speaking, we allow base manifolds to be iterated bundles with almost nonnegatively curved fibers. We first introduce some definitions which should make it easier to digest our results.

Let $\mathcal{S}_{r,e}$ be the class of smooth manifolds such that each $E \in \mathcal{S}_{r,e}$ admits a smooth family of Riemannian metrics $g_t^E$ with $(E, g_t^E) \in \mathcal{M}$, and $|K_{g_t^E}| \leq L_e/t^e$ for some constant $L_e$. In these notations Theorem 9.2 says that the total space of any vector bundle over a compact
manifold in \( \mathcal{S}_{r,e} \) lies in \( \mathcal{M}_r \). By reparametrizing, one sees that \( \mathcal{S}_{r,0} = \mathcal{S}_{q,0} \) for any \( r, q > 0 \), so we denote \( \mathcal{S}_{\infty,0} = \cap_{r>0} \mathcal{S}_{r,0} = \cup_{r>0} \mathcal{S}_{r,0} \). Thus if \( E \in \mathcal{S}_{\infty,0} \), then for each \( r \), \( E \) has a family of metrics \( g_t^{E,r} \) with \( (E, g_t^{E,r}) \in \mathcal{S}_{r,0} \).

Given a collection \( g_\alpha \) of Riemannian metrics on a manifold \( E \), we refer to the group \( \cap_\alpha \text{Iso}(E, g_\alpha) \) as the symmetry group of \( (E, g_\alpha) \) and denote it by \( \text{Sym}(E, g_\alpha) \). In particular, if \( (E, g_t^{E}) \in \mathcal{S}_{r,e} \), then \( \text{Sym}(E, g_t^{E}) \) is \( \cap_\alpha \text{Iso}(E, g_t^{E}) \), and if \( (E, g_t^{E,r}) \in \mathcal{S}_{\infty,e} \), then \( \text{Sym}(E, g_t^{E,r}) \) is \( \cap_{t,r} \text{Iso}(E, g_t^{E,r}) \).

**Example 10.1.**

(1) If \( \text{Ric}(E, g) \geq 0 \), then by Example 7.1, \( \text{Sym}(E, g) = \text{Iso}(E, g) \) is the symmetry group of the family consisting of the metric \( g \), and \( (E, g) \in \mathcal{S}_{\infty,0} \).

(2) Let \( E \) be the total space of a fiber bundle \( F \to E \to B \) where \( B \) is a compact manifold in \( \mathcal{S}_{\infty,0} \), \( F \) is a compact flat manifold, and the structure group of the bundle lies in \( \text{Iso}(F) \). Then \( E \in \mathcal{S}_{\infty,0} \) by Lemma 9.1(B) where we equip \( E \) with the metric as in Example 2.1.

(3) Let \( E \) be the total space of a flat fiber bundle \( F \to E \to B \) where \( (B, g_t^{B,r}) \in \mathcal{S}_{\infty,0} \), \( (F, g_t^{F,r}) \in \mathcal{S}_{\infty,0} \), and the structure group of the bundle lies in \( \text{Sym}(F, g_t^{F,r}) \). Equip \( E \) with the metric as in Example 2.3. If \( B, F \) are compact, then by Lemma 9.1(C), \( E \in \mathcal{S}_{\infty,0} \).

(4) Let \( E \) be the total space of a fiber bundle \( F \to E \to B \) where \( (B, g_t^B) \in \mathcal{S}_{q,b} \), \( (F, g_t^F) \in \mathcal{S}_{r,f} \), and the structure group of the bundle lies in \( \text{Sym}(F, g_t^F) \). Equip \( E \) with the metric as in Example 2.3. If \( B, F \) are compact, then by Lemma 9.1(C), \( E \in \mathcal{S}_{p,e} \) for some \( p, e \).

(5) Let \( E \) be the total space of a flat fiber bundle \( F \to E \to B \) where \( (B, g_t^B) \in \mathcal{S}_{q,b} \), \( (F, g_t^F) \in \mathcal{S}_{r,f} \), and the structure group of the bundle lies in \( \text{Sym}(F, g_t^F) \). Equip \( E \) with the metric as in Example 2.3. If \( B, F \) are compact, then by Lemma 9.1(C), \( E \in \mathcal{S}_{p,e} \) for some \( p, e \).

**Remark 10.2.** As one may suspect, the symmetry group is often discrete so that any bundle with the structure group being the symmetry group of the fiber is almost the product. However, here is an example with a large symmetry group. Let \( (B, g_t^B) \in \mathcal{S}_{r,e} \) and let \( G \) be a compact Lie group with a biinvariant metric. Then the total space \( E \) of any principal \( G \)-bundle over \( B \), equipped with the family of metrics \( g_t^E \) as in Example 2.1, has the symmetry group containing \( G \). Furthermore, if \( g_t^{E,r} \) is any reparametrization of \( g_t^E \), then of course \( G \leq \text{Iso}(g_t^{E,r}) \).

11 Fiber bundles with no nonnegative Ricci curvature

In this section we justify the claim made in the introduction that if \( E \) is a compact manifold with \( \text{Ric}(E) \geq 0 \) which fibers over a torus \( T \), then the pullback of the bundle \( E \to T \) to a finite cover of \( T \) has a section. Our only tool is the fact that a finite cover of \( E \) is homeomorphic to the product of a simply-connected manifold and a torus [CG72]. In fact, we prove a more general result as follows.

Let \( E \) be a manifold such that a finite cover \( p: \tilde{E} \to E \) is homeomorphic to the product of a simply-connected manifold \( C \), and a \( k \)-torus \( T^k \). Fix an inclusion \( i: T^k \to \tilde{E} \). Assume that there is a torus \( T \), and a homomorphism \( \phi: \pi_1(E) \to \pi_1(T) \) with finite cokernel.
(Such a homomorphism always exists if \(E\) is a compact manifold that fibers over a torus thanks to the homotopy exact sequence of the fibration).

Since \(T\) is aspherical, \(\phi\) is induced by a continuous map \(f: E \to T\). The composition \(\phi \circ p \circ i_*: \pi_1(T^k) \to \pi_1(T)\) also has finite cokernel, so there is a finite cover \(\tilde{T} \to T\) and a lift \(q: T^k \to \tilde{T}\) of \(f \circ p \circ i\) such that \(q\) is \(\pi_1\)-surjective. Any surjection of free abelian groups has a section, and since \(T^k\) is aspherical, \(q\) has a homotopy section. Composing the homotopy section with \(p \circ i\), we get a map \(s: \tilde{T} \to E\) such that \(f \circ s\) is homotopic to \(\pi\).

Replacing \(E\) by a homotopy equivalent space \(E'\), we can think of \(f: E \to T\) as a Serre fibration \(f': E' \to T\). Then the pullback of the fibration via \(\pi\) has a homotopy section induced by \(s\), and by the covering homotopy theorem \(f'\) has a section. Similarly, if \(E \to T\) is a fiber bundle to begin with, then its \(\pi\)-pullback has a section.

**Example 11.1.**

1. Let \(E \to T\) be a principal \(G\)-bundle over a torus \(T\) where \(G\) is a compact Lie group. If \(\text{Ric}(E) \geq 0\), then \(E \to T\) becomes trivial in a finite cover because any principal bundle with a section is trivial. Using obstruction theory as in [BK01, Section 4], it is easy to find \(G\)-bundles over \(T\) which do not become trivial in a finite cover.

2. Let \(E \to T\) be a sphere bundle over a torus with a nonzero rational Euler class. Then since finite covers induce injective maps on rational cohomology, \(E\) does not admit a metric with \(\text{Ric} \geq 0\) (see [BK01, Section 6] for related results).

3. Let \(E\) be an iterated fiber bundle, i.e. \(E = E_0 \to E_1 \to \cdots \to E_n\) where \(E_{i-1} \to E_i\) is a fiber bundle for each \(i\). Assume that \(E\) is a compact manifold, and \(E_n = T\) is a torus. Since the composition of all the bundle projections \(E \to T\) is a fiber bundle, it has a section if \(\text{Ric}(E) \geq 0\), in which case the bundle \(E_i \to T\) has a section for each \(i\). Thus, if \(E_{n-1} \to E_n = T\) is a bundle which does not have a section in a finite cover (e.g. as in (1) or (2)), then \(E\) admits no metric with \(\text{Ric} \geq 0\).

**References**


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